Advanced Mathematical Methods

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1 Linear Algebra

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WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Outline: Linear Algebra

$\underset{n \times n}{A}$ quadratic /square

1.8 Eigenvalues and eigenvectors

1.9 Quadratic forms and sign definitness ← symmetric matrix

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 21: Eigenvalues and Eigenvectors
  https://www.youtube.com/watch?v=lXNXrLcoerU
- Lecture 22: Powers of a square matrix and Diagonalization
  https://www.youtube.com/watch?v=13r9QY6cmjc
- Lecture 26: Symmetric matrices and positive definiteness
  https://www.youtube.com/watch?v=umt6BB1nJ4w
- Lecture 27: Positive definite matrices and minima – Quadratic forms
  https://www.youtube.com/watch?v=vF7eyJ2g3kU

1.8 Eigenvalues and eigenvectors

Assume a scalar λ exists such that $\underset{\sim}{b}$

$$Ax - \lambda x = 0$$

$$\underbrace{(A - \lambda I)}_{B} x = 0$$

$$\underset{n\times n}{A}\,\underset{n\times 1}{x} = \underset{1\times 1}{\lambda}\,\underset{n\times 1}{x}$$

λ: eigenvalue

x: eigenvector

x eigenvectors of A

λ eigenvalues of A

$Bx = 0$

homogenous LES

Find λ via the homogenous linear equation system

$$\underbrace{(\boldsymbol{A} - \lambda \boldsymbol{I})}_{= B}\boldsymbol{x} = 0$$

unique solution

$\det(B) \neq 0 \Leftrightarrow rk(B) = n \Leftrightarrow B^{-1}$ exists

$\Downarrow$

$\underset{n\times 1}{x} = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ trivial solution

infinitely many

$\boxed{\det(B) = 0} \Leftrightarrow B^{-1}$ does not exist

1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- in any case a solution does exist;
- if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) \neq 0$, then $\bar{\boldsymbol{x}} = 0$ is the trivial solution;
- only if $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$ there is a non-trivial solution.

ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad B = A - \lambda I_2$

$B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$

$\det(B = |B| \overset{!}{=} 0$

solve this equation

$\det(B) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)\cdot(2-\lambda) - 1\cdot 0 \overset{!}{=} 0$

roots

characteristic equation

$= 6 - 2\lambda - 3\lambda + \lambda^2$

$= 6 - 5\lambda + \lambda^2 \overset{!}{=} 0$

$\lambda_1 = 3 \quad \lambda_2 = 2$

1.8 Eigenvalues and eigenvectors

$Ax = \lambda x$ $\qquad (A - \lambda I)x = 0$

How do I find λ and x ?

① find eigenvalues : λ roots of the characteristic equation

$n=2$

$|A - \lambda I| = 0 \rightarrow \lambda^2 - a\lambda + b \stackrel{!}{=} 0$

$\rightarrow$ solve for $\lambda_1, \dots, \lambda_n$

② for each λ_i, find the corresponding eigenvectors x_i

$Ax_i = \lambda_i x_i \rightarrow$ solve a LES

$\underbrace{(A - \lambda_i I)}_{\text{singular}} x_i = 0 \rightarrow$ homogenous LES

infinitely many solutions:

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| \overset{!}{\doteq} 0 \quad \Longleftrightarrow \quad (-1)^n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \overset{!}{\doteq} 0$$

$\lambda_1 = 3$ $\quad \lambda_2 = 2$

$$\begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 1\cdot 0 \overset{!}{\doteq} 0$$

$$= \underbrace{6}_{\det(A)} - \underbrace{5}_{tr(A)}\lambda + \lambda^2 \overset{!}{\doteq} 0$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$\det(A) = 3\cdot 2 - 1\cdot 0 = 6$

$tr(A) = 3 + 2 = 5$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$-Matrix $\boldsymbol{A}$ we can calculate the respective eigenvector $\boldsymbol{x}_i \neq 0$ solving the homogenous linear equation system

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x}_i = 0 \,. \tag{1}$$

$\det(A) = 6 = \lambda_1 \cdot \lambda_2 = 3 \cdot 2 = 6$

$tr(A) = 5 = \lambda_1 + \lambda_2 = 3 + 2 = 5$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors $\boldsymbol{x}_i$.

Mitternachtsformel (MNF)

2x2 quadratic formula:

$$a x^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ex: $\lambda^2 - 5\lambda + 6 = 0$ $\quad a = 1,\ b = -5,\ c = 6$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2} \quad \nearrow \lambda_1 = 3 \quad \searrow \lambda_2 = 2$$

complex solution: $\sqrt{-(\cdot)}$

$i \equiv \sqrt{-1}$ $\quad$ ex: $\lambda_1 = 5 + 2i$ $\quad \lambda_2 = 5 - 2i$

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

1 $$|\boldsymbol{A} - \lambda \boldsymbol{I}| \; = \; 0 \quad \Longleftrightarrow \quad (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots + \alpha_1 \lambda + \alpha_0 \; = \; 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$-Matrix $\boldsymbol{A}$ we can calculate the respective eigenvector $\boldsymbol{x}_i \neq 0$ solving the homogenous linear equation system

2 $$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x}_i = 0 \,. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors $\boldsymbol{x}_i$.

1.8 Eigenvalues and eigenvectors

② solve for eigenvectors : $\lambda_1 = 3$ λ_2 $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$\lambda_1 = 3$: $Ax_1 = \lambda_1 x_1 = 3x_1$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \rightarrow \begin{aligned} 3x + \underset{}{\overset{=0}{y}} &= 3x \\ 2y &= 3y \end{aligned}$$

$3x = 3x \rightarrow x = a$

$\rightarrow y = 0$

infinitely many solutions

$$x_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

normalized eigenvector length = 1

$$\|x_1\| = \sqrt{a^2 + 0^2} = a \underbrace{\overset{!}{=} 1}$$

$$\tilde{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_1 \cdot \left(\frac{1}{\|x_1\|} \right)$$

1.8 Eigenvalues and eigenvectors

$\lambda_2 = 2$:

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \rightarrow \begin{matrix} 3x + \overset{b}{y} = 2x \\ 2y = 2y \end{matrix} \quad \begin{matrix} x = -b \\ \rightarrow y = b \end{matrix}$$

$$x_2 = \begin{bmatrix} -b \\ b \end{bmatrix}$$

normalized version

$$\|x_2\| = \sqrt{(-b)^2 + b^2} = \sqrt{2b^2} = b\sqrt{2} \overset{!}{=} 1 \rightarrow b = \frac{1}{\sqrt{2}}$$

$$\tilde{x}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

1.8 Eigenvalues and eigenvectors

$T = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \end{bmatrix}$

$\boldsymbol{A}$ und $\boldsymbol{B}$ (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix $\boldsymbol{C}$ exists, such that

$$\boldsymbol{B} = \boldsymbol{C}^{-1} \boldsymbol{A} \, \boldsymbol{C} \, .$$

$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

A and B are similar if

1. same characteristic equation
2. same eigenvalues
3. same determinant

C needs to be invertible

ex: $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Special case: symmetric matrices

For a symmetric $(n \times n)$-matrix $\boldsymbol{A}$ it holds that the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ with $j = 1, \ldots, n$ have the property

1. $\tilde{\boldsymbol{x}}_j' \tilde{\boldsymbol{x}}_j = 1$ for any j and
2. $\tilde{\boldsymbol{x}}_i' \tilde{\boldsymbol{x}}_j = 0$ for all $i \neq j$

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda = \underset{\text{diagonal}}{D} = T^{-1} A T$

1.8 Eigenvalues and eigenvectors

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Special case: **symmetric matrices**

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1. $\tilde{\boldsymbol{x}}_j' \tilde{\boldsymbol{x}}_j = 1$ for all j and   length 1
2. $\tilde{\boldsymbol{x}}_i' \tilde{\boldsymbol{x}}_j = 0$ for all $i \neq j$.

orthogonal

T

matrix of eigenvectors

orthonormal

$T^{-1} = T^T$

1.8 Eigenvalues and eigenvectors

Principle axis theorem /spectral decomposition

collecting the normalized eigenvectors $\tilde{\boldsymbol{x}}_j$ $(j = 1, \ldots, n)$ in a new matrix $\boldsymbol{T} = [\tilde{\boldsymbol{x}}_1 \cdots \tilde{\boldsymbol{x}}_n]$ with the property $\boldsymbol{T}^{-1} = \boldsymbol{T}'$ yields the diagonalization of $\boldsymbol{A}$ as follows:

$$A = T \Lambda T^{-1}$$

$$\boldsymbol{D} = \underbrace{\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}}_{\text{symmetric matrices}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} = \Lambda$$

ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad T = \begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\boldsymbol{A}| = \prod_{i=1}^{n} \lambda_i$.

2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.

$\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

3) The matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ have the same eigenvalues.

at least one $\lambda_i = 0$

4) For a non-singular matrix $\boldsymbol{A}$ all eigenvalues λ we have: $|\boldsymbol{A}^{-1} - \frac{1}{\lambda}\boldsymbol{I}| = 0$.

5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

$\det(A) = 0$

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$$A^T = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$k = -1 \quad \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{bmatrix}$$

$$A^k = T \Lambda^k T^{-1}$$

$$k = 2 \quad \Lambda^2 = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$$

$$A^2 = A \cdot A = \underbrace{T \Lambda T^{-1}}_{A} \underbrace{T^{-1} T}_{I} \underbrace{\Lambda T^{-1}}_{A}$$

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5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

$T^{-1} = T^{T}$ orthonormal

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

6) The rank of a symmetric matrix $\boldsymbol{A}$ is equal to the number of eigenvalues different from zero.
7) The sum of the eigenvalues is equal to the trace: $\mathrm{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} \lambda_i$.
8) It holds that the eigenvalues of $\boldsymbol{A}^k$ are λ_i^k for all $i = 1, \ldots, n$ as $\boldsymbol{A}^k = \boldsymbol{T}\boldsymbol{\Lambda}^k\boldsymbol{T}^{-1}$.
9) $\boldsymbol{A}$ has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

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$$\Lambda = T^{-1} A T$$

1.9 Quadratic forms and sign definitness

Definitions

$f(x,y) = xy^2 + x^3 + yx^2$

$\hookrightarrow$ form degree 3

$f(x) = 7x^7 + 2x^5$

degree 7

$f(x,y) = xy^6 \to 7$

- Degree of a polynomial
- Form of *n*th degree
- special case: quadratic form

degree = 2

$$Q(x_1, x_2) = a_{11}\underline{x_1^2} + 2a_{12}\underline{x_1 x_2} + a_{22}\underline{x_2^2} \quad \gtreqless 0$$

> 0 positive definite

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T A x \gtreqless 0$$

< 0 negative definite

$\gtreqless 0$ indefinite

1.9 Quadratic forms and sign definitness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

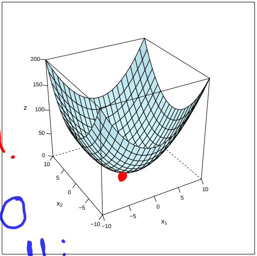
$$Q(x_1, x_2) = \underset{(1\times2)(2\times2)(2\times1)}{\mathrm{x}'\;\mathrm{A}\;\mathrm{x}} = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}\, x_i\, x_j$$

where $a_{ij} = a_{ji}$ and, thus,

with the symmetric coefficient matrix $\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$.

1.9 Quadratic forms and sign definitness

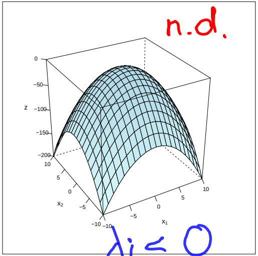
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



p.d.

$\lambda_i > 0$ for all i

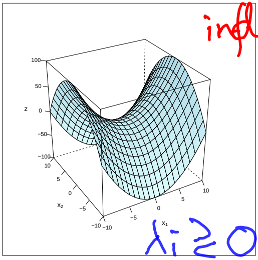
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



n.d.

$\lambda_i < 0$

Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



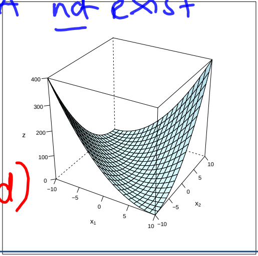
indefinit

$\lambda_i \gtrless 0$

$\lambda_i \geq 0$

A^{-1} not exist

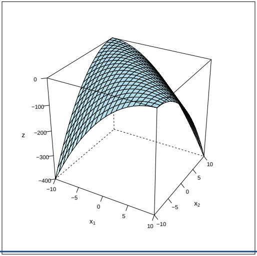
Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



≥ 0

positive semi definite (p.s.d)

Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



≤ 0

negative semi definite

$\lambda_i \leq 0$

1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = x'Ax > 0$	for all $x \neq 0$
positive semi-definite,	if $Q = x'Ax \geq 0$	for all x
negative definite,	if $Q = x'Ax < 0$	for all $x \neq 0$
negative semi-definite,	if $Q = x'Ax \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $x'Ax = x'Bx$ with $B = 0,5 \cdot (A + A')$, a symmetric matrix.

A quadratic but not symmetric

ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad B = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{bmatrix}$

1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = \mathrm{x}'\mathrm{Ax} > 0$	for all $\mathrm{x} \neq 0$
positive semi-definite,	if $Q = \mathrm{x}'\mathrm{Ax} \geq 0$	for all x
negative definite,	if $Q = \mathrm{x}'\mathrm{Ax} < 0$	for all $\mathrm{x} \neq 0$
negative semi-definite,	if $Q = \mathrm{x}'\mathrm{Ax} \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $\mathrm{x}'\mathrm{Ax} = \mathrm{x}'\mathrm{Bx}$ with $\mathrm{B} = 0,5 \cdot (\mathrm{A} + \mathrm{A}')$, a symmetric matrix.

1.9 Quadratic forms and sign definitness

The quadratic form $Q(\boldsymbol{x})$ is

- positive (negative) definite, if **all** eigenvalues of the matrix A are positive (negative): $\lambda_j > 0$ $(\lambda_j < 0)$ $\forall j = 1, 2, \ldots, n$;
- positive (negative) semi-definite, if **all** eigenvalues of the matrix A are non-negative (non-positive): $\lambda_j \geq 0$ $(\lambda_j \leq 0)$ $\forall j = 1, 2, \ldots, n$ and **at least one** eigenvalue is equal to zero;
- indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definitness

Properties of positive definite and positive semi-definite matrices

1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
2) If A is positive definite, then A^{-1} exists and is positive definite.
3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
4) If X is $n \times k$ and $rk(X) = k$, then $X'X$ is positive definite (and therefore non-singular).

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Relationship bt. A^{-1}, det(A), rk(A) and λ

A is invertible
nxn

- $\det(A) \neq 0$
- $rk(A) = n$ full rank
- n linearly independent rows/columns
- LES: $Ax = b$
  unique solution x
- λ_i are nonzero

A is singular

- $\det(A) = 0$
- $rk(A) < n$ rank deficient
- at least one $\lambda_i = 0$
- linear dependencies in A
- LES: $Ax = b$
  infinitely many
  $Ax = 0$
  no solution