EBERHARD KARLS UNIVERSITÄT TÜBINGEN

WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

Dr. Julie Schnaitmann

S414

Advanced Mathematical Methods

Exercises

WS 2023/24

Probability and Distribution Theory

Exercise 1 **Probability and Distribution Theory**

Given a continuous random variable X with:

$$f(x) = \begin{cases} 4ax & 0 \leq x < 1 \\ -ax + 0.5 & 1 \leq x \leq 5 \\ 0 & \text{else} \end{cases}$$

Determine the parameter a such that $f(x)$ is a density function of X. Calculate the corresponding distribution function and sketch it. Compute the expectation and the variance of X.

Exercise 2 **Probability and Distribution Theory**

The Federal Statistical Office assumes all values in the interval $2 \leq x \leq 3$ to be possible realizations of the random variable X : "Growth rate of the GDP". Moreover, the following function is assumed:

$$f(x) = \begin{cases} c \cdot (x-2) & 2 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

a) Determine c such that the function $f(x)$ is a density function of the random variable X.

b) Compute the distribution function of the random variable X.

c) Compute $P(X < 2.1)$ and $P(2.1 < X < 2.8)$.

d) Compute $P(-4 \leq X \leq 3|X \leq 2.1)$ and show that the events $\{-4 \leq X \leq 3\}$ and $\{X \leq 2.1\}$ are statistically independent.

e) Compute the expectation, median and the variance of X.

Exercise 3 **Probability and Distribution Theory**

Show the Markov - inequality:

$$P(X \geq c) \leq \frac{E[X]}{c}$$

for every positive value of c with X being strictly non-negative.

Exercise 4 Probability and Distribution Theory

		X: 1	2	3
Y	1	0.25	0.15	0.10
	2	0.10	0.15	0.25

a) Compute the expectation and the variance of X and Y.

b) Determine the conditional distributions of $X|Y = y$ and $Y|X = x$.

c) Determine the covariance and the correlation coefficient of X and Y.

d) Determine the variance of $X + Y$.

Exercise 5 Probability and Distribution Theory

The joint probability function of X and Y is given by:

$$f(x, y) = \begin{cases} e^{-2\lambda} \cdot \frac{\lambda^{x+y}}{x!y!} & x, y \in \{0, 1, \ldots\} \\ 0 & \text{else} \end{cases}$$

a) Determine the marginal distributions of X and Y.

b) Determine the conditional distributions of $X|Y = y$ and $Y|X = x$ and compare them to the marginal distributions.

c) Determine the covariance of X and Y.

Hint: $e^{\lambda} = \sum_y \dfrac{\lambda^y}{y!}$

Exercise 6 Probability and Distribution Theory

Suppose that x_u is the u percentile of the random variable X, that is, $F(x_u) = u$. Show that if $f(-x) = f(x)$, then $x_{1-u} = -x_u$

Exercise 7 Probability and Distribution Theory

If $X \sim N(1000, 400)$ find:

a) $P(X < 1024)$

b) $P(X < 1024|X > 961)$

c) $P(31 < \sqrt{X} < 32)$

Exercise 8 Probability and Distribution Theory

A fair coin is tossed three times and the random variable X equals the total number of heads. Find and sketch $F_X(x)$ and $f_X(x)$.

Exercise 9 Probability and Distribution Theory

The random variables X and Y are $N(\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho_{xy}) = N(3, 4, 1, 4, 0.5)$. Find $f(y|x)$ and $f(x|y)$.

Solution Exercise 1:

$a = \frac{1}{10}$

$\mathbb{E}[x] = 2$

$\text{var}(x) = 1.1666\bar{6}$

Solution Exercise 2:

a) $c = 2$

b)

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ x^2 - 4x + 4 & \text{for } 2 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}$$

c) $P(X < 2.1) = \underline{\underline{0.01}}$
$P(2.1 < X < 2.8) = \underline{\underline{0.63}}$

d)

$$P\left(-4 \leq X \leq 3 | X \leq\right) = \underline{\underline{1}}$$

The events $A = \{-4 \leq X \leq 3\}$ and $B = \{x \leq 2.1\}$ are independent if $P(B|A) = P(B)$.
We have: $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$
with $P(A) = P(-4 \leq X \leq 3) = 1$,
and $P(A|B) = P(-4 \leq X \leq 3 | x \leq 2.1) = 1$.
Hence: $P(A \cap B) = P(B|A) = P(B)$ q.e.d.

e)
- $\underline{\underline{\bar{x}[0.5] = 2 + \frac{1}{\sqrt{2}} = 2.7071.}}$
- $\mathbb{E}[x] = \underline{\underline{2.66\bar{6}}}$
- $\text{var}[x] = \underline{\underline{\frac{1}{18} = 0.05\bar{5}}}$

Solution Exercise 3:

$$\begin{aligned}
\mathbb{E}[x] &= \int_{-\infty}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx \\
\mathbb{E}[x] &> \int_{c}^{\infty} xf(x)dx \\
\mathbb{E}[x] &> c\int_{c}^{\infty} f(x)dx \\
&> cP(X \geq c) \\
\frac{\mathbb{E}[x]}{c} &= P(X \geq c) \ \text{ q.e.d.}
\end{aligned}$$

Solution Exercise 4:

a) $\mathbb{E}[x] = 2$
   $\mathbb{E}[x] = 1.5$

b) The conditional probability distributions:

		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	0.5	0.3	0.2
distribution $f(x\|y)$	$y = 2$	0.2	0.3	0.5
		$x = 1$	$x = 2$	$x = 3$
Conditional	$y = 1$	5/7	1/2	2/7
distribution $f(x\|y)$	$y = 2$	2/7	1/2	5/7

c) $\text{cov}[x, y] = \underline{\underline{0.15}}$

d) $\text{var}[x + y] = 1.25$

Solution Exercise 5:

a) $f(x) = \underline{\underline{e^{-\lambda}\frac{\lambda^x}{x!}}}$ and $f(y) = \underline{\underline{e^{-\lambda}\frac{\lambda^y}{y!}}}$

b) $f(x|y) = f(x)$

   $f(y|x) = f(y)$

c) $\text{cov}[x, y] = 0$

Solution Exercise 6:

If $f(x) = f(-x)$ then $\int_{-\infty}^{-x_u} f(z)dz = \int_{x_u}^{\infty} f(z)dz$.

From which follows that:

$$F(-x_u) = 1 - F(x_u) = 1 - u$$

Hence, $-x_u = x_{1-u}$ q.e.d.

Solution Exercise 7:

a) $P(X < 1024) = 0.8849$

b) $P(X < 1024|X > 961) = 0.8819$

c) $P(31 < \sqrt{x} < 32) = 0.8593$

Solution Exercise 8:

$$f_X(x) = 0.5^3 \binom{3}{x} = 0.5^3 \frac{3!}{x!(3-x)!}$$
$$F_X(x) = 0.5^3 \sum_{k=1}^{x} \binom{3}{k} = 0.5^3 \sum_{k=1}^{x} \frac{3!}{k!(3-k)!}$$

Solution Exercise 9:

$Y|X \sim N\left(\mu_y + \rho_{xy}\frac{\sigma_y}{\sigma_x}(x - \mu_x); \sigma_y^2(1 - \rho_{xy}^2)\right)$

$X|Y \sim N\left(\mu_x + \rho_{xy}\frac{\sigma_x}{\sigma_y}(y - \mu_y); \sigma_x^2(1 - \rho_{xy}^2)\right)$